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On some delay nonlinear integral inequalities in two independent variables

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Abstract
The purpose of this paper is to generalize some integral inequalities in two independent variables with delay which can be used as handy tools in the study of certain partial differential equations and integral equations with delay. An application is given to illustrate the usefulness of our results.

Keywords: retarded integral inequalities; two independent variables; differential and integral equations with time delay; nondecreasing functions

1 Introduction

The integral inequalities which provide explicit bounds on unknown functions play an important role in the development of the theory of differential and integral equations. The Gronwall-Bellman inequality and its various linear and nonlinear generalizations are crucial in the discussion of the existence, uniqueness, continuation, boundedness, oscillation and stability, and other qualitative properties of solutions of differential and integral equations. The literature on such inequalities and their applications is vast; see [1–7] and the references given therein.

In [8] Ferreira and Torres, have discussed the following useful nonlinear retarded integral inequality:

$$\phi\bigl(u(t)\bigr) \le c(t) + \int_0^{\alpha(t)} \bigl[f(t,s)\eta\bigl(u(s)\bigr)\omega\bigl(u(s)\bigr) + g(t,s)\eta\bigl(u(s)\bigr)\bigr]\,ds.$$

Motivated by the results obtained in [8, 9] and [10] we establish a general two independent variables retarded version which can be used as a tool to study the boundedness of solutions of differential and integral equations.

2 Main results

In what follows, R denotes the set of real numbers, $R_+ = [0,+\infty)$, $I_1 = [0,M]$, $I_2 = [0,N]$ are the given subsets of R, and $\Delta = I_1 \times I_2$. $C^i(A,B)$ denotes the class of all i times continuously differentiable functions defined on a set A with range in the set B ($i = 1,2,\ldots$) and $C^0(A,B) = C(A,B)$.

Lemma 2.1 *Let $u(x,y), f(x,y), \sigma(x,y) \in C(\Delta,R_+)$ and $a(x,y) \in C(\Delta,R_+)$ be nondecreasing with respect to $(x,y) \in \Delta$, let $\alpha \in C^1(I_1,I_1)$, $\beta \in C^1(I_2,I_2)$ be nondecreasing with $\alpha(x) \le x$ on*

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I_1, $\beta(y) \leq y$ on I_2. Further let $\psi, \omega \in C(R_+, R_+)$ be nondecreasing functions with $\{\psi, \omega\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \psi(u) = +\infty$. If $u(x,y)$ satisfies

$$\psi\big(u(x,y)\big) \leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t)f(s,t)\omega\big(u(s,t)\big)\,dt\,ds \tag{2.1}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \psi^{-1}\left\{G^{-1}\left[G\big(a(x,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t)f(s,t)\,dt\,ds\right]\right\} \tag{2.2}$$

for $0 \leq x \leq x_1$, $0 \leq y \leq y_1$, where

$$G(v) = \int_{v_0}^{v} \frac{ds}{\omega(\psi^{-1}(s))}, \quad v \geq v_0 > 0, \qquad G(+\infty) = \int_{v_0}^{+\infty} \frac{ds}{\omega(\psi^{-1}(s))} = +\infty \tag{2.3}$$

and $(x_1, y_1) \in \Delta$ is chosen so that $(G(a(x,y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds) \in \mathrm{Dom}(G^{-1})$.

Theorem 2.2 *Let u, a, f, α, and β be as in Lemma* 2.1. *Let $\sigma_1(x,y), \sigma_2(x,y) \in C(\Delta, R_+)$. Further $\psi, \omega, \eta \in C(R_+, R_+)$ be nondecreasing functions with $\{\psi, \omega, \eta\}(u) > 0$ for $u > 0$, and* $\lim_{u\to+\infty} \psi(u) = +\infty$.

(A$_1$) *If $u(x,y)$ satisfies*

$$\psi\big(u(x,y)\big) \leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left[f(s,t)\omega\big(u(s,t)\big) + \int_0^{s} \sigma_2(\tau,t)\omega\big(u(\tau,t)\big)\,d\tau\right]dt\,ds \tag{2.4}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \psi^{-1}\left\{G^{-1}\left(p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right)\right\} \tag{2.5}$$

for $0 \leq x \leq x_1$, $0 \leq y \leq y_1$, where G is defined by (2.3) *and*

$$p(x,y) = G\big(a(x,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left(\int_0^{s} \sigma_2(\tau,t)\,d\tau\right)dt\,ds \tag{2.6}$$

and $(x_1, y_1) \in \Delta$ is chosen so that $(p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds) \in \mathrm{Dom}(G^{-1})$.

(A$_2$) *If $u(x,y)$ satisfies*

$$\psi\big(u(x,y)\big) \leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) + \int_0^{s} \sigma_2(\tau,t)\omega\big(u(\tau,t)\big)\,d\tau\right]dt\,ds \tag{2.7}$$

for $(x,y) \in \Delta$, then

$$u(x,y) \leq \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[F\big(p(x,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right]\right)\right\} \tag{2.8}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$, *where* G *and* p *are as in* (A_1), *and*

$$F(v) = \int_{v_0}^{v} \frac{ds}{\eta(\psi^{-1}(G^{-1}(s)))}, \quad v \ge v_0 > 0, \qquad F(+\infty) = +\infty, \tag{2.9}$$

and $(x_1, y_1) \in \Delta$ *is chosen so that* $[F(p(x,y)) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds] \in \mathrm{Dom}(F^{-1})$.

(A_3) *If* $u(x,y)$*satisfies*

$$\begin{aligned} \psi\big(u(x,y)\big) \le{} & a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\bigg[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) \\ & + \int_0^{s} \sigma_2(\tau,t)\omega\big(u(\tau,t)\big)\eta\big(u(\tau,t)\big)\,d\tau\bigg]\,dt\,ds \end{aligned} \tag{2.10}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \le \psi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right]\right)\right\} \tag{2.11}$$

for $0 \le x \le x_1$, $0 \le y \le y_1$ *where*

$$p_0(x,y) = F\big(G\big(a(x,y)\big)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left(\int_0^{s} \sigma_2(\tau,t)\,d\tau\right)dt\,ds$$

and $(x_1, y_1) \in \Delta$ *is chosen so that* $[p_0(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds] \in \mathrm{Dom}(F^{-1})$.

The proof of the theorem will be given in the next section.

Remark 2.3 If we take $\sigma_2(x,y) = 0$, then Theorem 2.2(A_1) reduces to Lemma 2.1.

Corollary 2.4 *Let the functions* u, f, σ_1, σ_2, a, α, *and* β *be as in Theorem* 2.2. *Further* $q > p > 0$ *are constants.*

(B_1) *If* $u(x,y)$ *satisfies*

$$\begin{aligned} u^p(x,y) \le{} & a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\bigg[f(s,t)u^p(s,t) \\ & + \int_0^{s} \sigma_2(\tau,t)u^p(\tau,t)\,d\tau\bigg]\,dt\,ds \end{aligned} \tag{2.12}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \le \big(a(x,y)\big)^{\frac{1}{p}} \exp\left(\frac{1}{p}\int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left[f(s,t) + \int_0^{s} \sigma_2(\tau,t)\,d\tau\right]dt\,ds\right). \tag{2.13}$$

(B_2) *If* $u(x,y)$ *satisfies*

$$\begin{aligned} u^q(x,y) \le{} & a(x,y) + \frac{q}{q-p}\int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\bigg[f(s,t)u^p(s,t) \\ & + \int_0^{s} \sigma_2(\tau,t)u^p(\tau,t)\,d\tau\bigg]\,dt\,ds \end{aligned} \tag{2.14}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \left\{ p(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t)\,dt\,ds \right\}^{\frac{1}{q-p}}, \tag{2.15}$$

where

$$p(x,y) = \big(a(x,y)\big)^{\frac{q-p}{q}} + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t)\,d\tau \right) dt\,ds.$$

Corollary 2.5 *Let the functions* u, a, f, σ_1, σ_2, α, *and* β *be as in Theorem* 2.2. *Further* q, p, *and* r *are constants with* $p > 0$, $r > 0$ *and* $q > p + r$.

(C_1) *If* $u(x,y)$ *satisfies*

$$u^q(x,y) \leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) u^p(s,t) u^r(s,t) + \int_0^s \sigma_2(\tau,t) u^p(\tau,t)\,d\tau \right] dt\,ds \tag{2.16}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \left\{ \big[p(x,y)\big]^{\frac{q-p-r}{q-p}} + \frac{q-p-r}{q} \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t)\,dt\,ds \right\}^{\frac{1}{q-p-r}}, \tag{2.17}$$

where

$$p(x,y) = \big(a(x,y)\big)^{\frac{q-p}{q}} + \frac{q-p}{q} \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t)\,d\tau \right) dt\,ds.$$

(C_2) *If* $u(x,y)$ *satisfies*

$$u^q(x,y) \leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left[f(s,t) u^p(s,t) u^r(s,t) + \int_0^s \sigma_2(\tau,t) u^p(\tau,t) u^r(\tau,t)\,d\tau \right] dt\,ds \tag{2.18}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \left\{ p_0(x,y) + \frac{q-p-r}{q} \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) f(s,t)\,dt\,ds \right\}^{\frac{1}{q-p-r}}, \tag{2.19}$$

where

$$p_0(x,y) = \big(a(x,y)\big)^{\frac{q-p-r}{q}} + \frac{q-p-r}{q} \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \left(\int_0^s \sigma_2(\tau,t)\,d\tau \right) dt\,ds.$$

Theorem 2.6 *Let* u, f, σ_1, σ_2, a, α, β, ψ, ω, *and* η *be as in Theorem* 2.2. *If* $u(x,y)$ *satisfies*

$$\psi\big(u(x,y)\big) \leq a(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t) \eta\big(u(s,t)\big) \times \left[f(s,t) \omega\big(u(s,t)\big) + \int_0^s \sigma_2(\tau,t)\,d\tau \right] dt\,ds \tag{2.20}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \psi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1\bigl(p_1(x,y)\bigr) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right]\right)\right\} \tag{2.21}$$

for $0 \leq x \leq x_2$, $0 \leq y \leq y_2$, *where*

$$G_1(\nu) = \int_{\nu_0}^{\nu} \frac{ds}{\eta(\psi^{-1}(s))}, \quad \nu \geq \nu_0 > 0, \qquad G_1(+\infty) = \int_{\nu_0}^{+\infty} \frac{ds}{\eta(\psi^{-1}(s))} = +\infty, \tag{2.22}$$

$$F_1(\nu) = \int_{\nu_0}^{\nu} \frac{ds}{\omega[\psi^{-1}(G_1^{-1}(s))]}, \quad \nu \geq \nu_0 > 0, \qquad F_1(+\infty) = +\infty, \tag{2.23}$$

$$p_1(x,y) = G_1\bigl(a(x,y)\bigr) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)\left(\int_0^s \sigma_2(\tau,t)\,d\tau\right)dt\,ds, \tag{2.24}$$

and $(x_2,y_2) \in \Delta$ *is chosen so that* $[F_1(p_1(x,y)) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds] \in \mathrm{Dom}(F_1^{-1})$.

Theorem 2.7 *Let* $u, f, \sigma_1, \sigma_2, a, \alpha, \beta, \psi$, *and* ω *be as in Theorem* 2.2, *and* $p > 0$ *a constant. If* $u(x,y)$ *satisfies*

$$\begin{aligned}\psi\bigl(u(x,y)\bigr) \leq a(x,y) &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)u^p(s,t)\\ &\times\left[f(s,t)\omega\bigl(u(s,t)\bigr) + \int_0^s \sigma_2(\tau,t)\,d\tau\right]dt\,ds\end{aligned} \tag{2.25}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq \psi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1\bigl(p_1(x,y)\bigr) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right]\right)\right\} \tag{2.26}$$

for $0 \leq x \leq x_2$, $0 \leq y \leq y_2$, *where*

$$G_1(\nu) = \int_{\nu_0}^{\nu} \frac{ds}{[\psi^{-1}(s)]^p}, \quad \nu \geq \nu_0 > 0, \qquad G_1(+\infty) = \int_{\nu_0}^{+\infty} \frac{ds}{[\psi^{-1}(s)]^p} = +\infty \tag{2.27}$$

and F_1, p_1 *are as in Theorem* 2.6 *and* $(x_2,y_2) \in \Delta$ *is chosen so that*

$$\left[F_1\bigl(p_1(x,y)\bigr) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right] \in \mathrm{Dom}\bigl(F_1^{-1}\bigr).$$

Remark 2.8 The inequality established in Theorem 2.7 generalizes Theorem 1 of [10] (with $p = 1$, $a(x,y) = b(x) + c(x)$, $\sigma_1(s,t)f(s,t) = h(s,t)$, and $\sigma_1(s,t)(\int_0^s \sigma_2(\tau,t)\,d\tau) = g(s,t)$).

Corollary 2.9 *Let* $u, f, \sigma_1, \sigma_2, a, \alpha, \beta$, *and* ω *be as in Theorem* 2.2 *and* $q > p > 0$ *be constants. If* $u(x,y)$ *satisfies*

$$\begin{aligned}u^q(x,y) \leq a(x,y) &+ \frac{p}{p-q}\int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)u^p(s,t)\\ &\times\left[f(s,t)\omega\bigl(u(s,t)\bigr) + \int_0^s \sigma_2(\tau,t)\,d\tau\right]dt\,ds\end{aligned} \tag{2.28}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \le \left\{ F_1^{-1}\left[F_1\big(p_1(x,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds \right] \right\}^{\frac{1}{q-p}} \tag{2.29}$$

for $0 \le x \le x_2$, $0 \le y \le y_2$, *where*

$$p_1(x,y) = \big[a(x,y)\big]^{\frac{q-p}{q}} + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\left(\int_0^s \sigma_2(\tau,t)\,d\tau \right) dt\,ds$$

and F_1 *is defined in Theorem* 2.6.

Remark 2.10 Setting $a(x,y) = b(x) + c(x)$, $\sigma_1(s,t)f(s,t) = h(s,t)$, and $\sigma_1(s,t)(\int_0^s \sigma_2(\tau,t)\,d\tau) = g(s,t)$ in Corollary 2.9 we obtain Theorem 1 of [11].

Remark 2.11 Setting $a(x,y) = c^{\frac{p}{p-q}}$, $\sigma_1(s,t)f(s,t) = h(t)$, and $\sigma_1(s,t)(\int_0^s \sigma_2(\tau,t)\,d\tau) = g(t)$ and keeping y fixed in Corollary 2.9, we obtain Theorem 2.1 of [12].

3 Proof of theorems

Proof of Lemma 2.1 First we assume that $a(x,y) > 0$. Fixing an arbitrary $(x_0,y_0) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$z(x,y) = a(x_0,y_0) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t)f(s,t)\omega\big(u(s,t)\big)\,dt\,ds$$

for $0 \le x \le x_0 \le x_1$, $0 \le y \le y_0 \le y_1$, then $z(0,y) = z(x,0) = a(x_0,y_0)$ and

$$u(x,y) \le \psi^{-1}\big(z(x,y)\big), \tag{3.1}$$

and then we have

$$\begin{aligned}
\frac{\partial z(x,y)}{\partial x} &= \alpha'(x) \int_0^{\beta(y)} \sigma\big(\alpha(x),t\big)f\big(\alpha(x),t\big)\omega\big(u\big(\alpha(x),t\big)\big)\,dt \\
&\le \alpha'(x) \int_0^{\beta(y)} \sigma\big(\alpha(x),t\big)f\big(\alpha(x),t\big)\omega\big(\psi^{-1}\big(z\big(\alpha(x),t\big)\big)\big)\,dt \\
&\le \omega\big(\psi^{-1}\big(z\big(\alpha(x),\beta(y)\big)\big)\big)\alpha'(x) \int_0^{\beta(y)} \sigma\big(\alpha(x),t\big)f\big(\alpha(x),t\big)\,dt
\end{aligned}$$

or

$$\frac{\frac{\partial z(x,y)}{\partial x}}{\omega(\psi^{-1}(z(x,y)))} \le \alpha'(x) \int_0^{\beta(y)} \sigma\big(\alpha(x),t\big)f\big(\alpha(x),t\big)\,dt.$$

Keeping y fixed, setting $x = s$, integrating the last inequality with respect to s from 0 to x, and making the change of variable $s = \alpha(x)$ we get

$$\begin{aligned}
G\big(z(x,y)\big) &\le G\big(z(0,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t)f(s,t)\,dt\,ds \\
&\le G\big(a(x_0,y_0)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t)f(s,t)\,dt\,ds.
\end{aligned}$$

Since $(x_0, y_0) \in \Delta$ is chosen arbitrary,

$$z(x,y) \leq G^{-1}\left[G\big(a(x,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma(s,t) f(s,t)\,dt\,ds\right].$$

So from the last inequality and (3.1) we obtain (2.2). If $a(x,y) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ and subsequently let $\epsilon \to 0$. □

Proof of Theorem 2.2 (A_1) By the same steps of the proof of Lemma 2.1 we can obtain (2.5), with suitable changes.

(A_2) Assume that $a(x,y) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z(x,y) = a(x_0,y_0) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\Big[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big) \\ + \int_0^s \sigma_2(\tau,t)\omega\big(u(\tau,t)\big)\,d\tau\Big]\,dt\,ds \end{aligned}$$

for $0 \leq x \leq x_0 \leq x_1$, $0 \leq y \leq y_0 \leq y_1$, then $z(0,y) = z(x,0) = a(x_0,y_0)$ and

$$\begin{aligned} & u(x,y) \leq \psi^{-1}\big(z(x,y)\big), \qquad (3.2) \\ & \frac{\partial z(x,y)}{\partial x} = \alpha'(x) \int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\Big[f\big(\alpha(x),t\big)\omega\big(u\big(\alpha(x),t\big)\big)\eta\big(u\big(\alpha(x),t\big)\big) \\ & \qquad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\omega\big(u(\tau,t)\big)\,d\tau\Big]\,dt \\ & \quad \leq \alpha'(x) \int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\Big[f\big(\alpha(x),t\big)\omega\big(\psi^{-1}\big(z\big(\alpha(x),t\big)\big)\big)\eta\big(\psi^{-1}\big(z\big(\alpha(x),t\big)\big)\big) \\ & \qquad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\omega\big(\psi^{-1}\big(z(\tau,t)\big)\big)\,d\tau\Big]\,dt \\ & \quad \leq \alpha'(x) \cdot \omega\big(\psi^{-1}\big(z\big(\alpha(x),\beta(y)\big)\big)\big) \\ & \qquad \times \int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\Big[f\big(\alpha(x),t\big)\eta\big(\psi^{-1}\big(z\big(\alpha(x),t\big)\big)\big) + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\Big]\,dt \end{aligned}$$

then

$$\begin{aligned} \frac{\frac{\partial z(x,y)}{\partial x}}{\omega(\psi^{-1}(z(x,y)))} \leq \alpha'(x) \int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\Big[f\big(\alpha(x),t\big)\eta\big(\psi^{-1}\big(z\big(\alpha(x),t\big)\big)\big) \\ + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\Big]\,dt. \end{aligned}$$

Keeping y fixed, setting $x = s$ integrating the last inequality with respect to s from 0 to x, and making the change of variable $s = \alpha(x)$ we get

$$\begin{aligned} G\big(z(x,y)\big) \leq G\big(z(0,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\Big[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big) \\ + \int_0^s \sigma_2(\tau,t)\,d\tau\Big]\,dt\,ds \end{aligned}$$

$$\begin{aligned}&\leq G\big(a(x_0,y_0)\big)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\bigg[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)\\&\quad+\int_0^s\sigma_2(\tau,t)\,d\tau\bigg]\,dt\,ds.\end{aligned}$$

Since $(x_0,y_0)\in\Delta$ is chosen arbitrarily, the last inequality can be rewritten as

$$G\big(z(x,y)\big)\leq p(x,y)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)\,dt\,ds. \tag{3.3}$$

Since $p(x,y)$ is a nondecreasing function, an application of Lemma 2.1 to (3.3) gives us

$$z(x,y)\leq G^{-1}\bigg(F^{-1}\bigg[F\big(p(x,y)\big)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)\,dt\,ds\bigg]\bigg). \tag{3.4}$$

From (3.2) and (3.4) we obtain the desired inequality (2.8).

Now we take the case $a(x,y)=0$ for some $(x,y)\in\Delta$. Let $a_\epsilon(x,y)=a(x,y)+\epsilon$, for all $(x,y)\in\Delta$, where $\epsilon>0$ is arbitrary, then $a_\epsilon(x,y)>0$ and $a_\epsilon(x,y)\in C(\Delta,R_+)$ be nondecreasing with respect to $(x,y)\in\Delta$. We carry out the above procedure with $a_\epsilon(x,y)>0$ instead of $a(x,y)$, and we get

$$u(x,y)\leq\psi^{-1}\bigg\{G^{-1}\bigg(F^{-1}\bigg[F\big(p_\epsilon(x,y)\big)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)f(s,t)\,dt\,ds\bigg]\bigg)\bigg\},$$

where

$$p_\epsilon(x,y)=G\big(a_\epsilon(x,y)\big)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\bigg(\int_0^s\sigma_2(\tau,t)\,d\tau\bigg)\,dt\,ds.$$

Letting $\epsilon\to0^+$, we obtain (2.8).

(A_3) Assume that $a(x,y)>0$. Fixing an arbitrary $(x_0,y_0)\in\Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned}z(x,y)&=a(x_0,y_0)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\bigg[f(s,t)\omega\big(u(s,t)\big)\eta\big(u(s,t)\big)\\&\quad+\int_0^s\sigma_2(\tau,t)\omega\big(u(\tau,t)\big)\eta\big(u(\tau,t)\big)\,d\tau\bigg]\,dt\,ds\end{aligned}$$

for $0\leq x\leq x_0\leq x_1$, $0\leq y\leq y_0\leq y_1$, then $z(0,y)=z(x,0)=a(x_0,y_0)$, and

$$u(x,y)\leq\psi^{-1}\big(z(x,y)\big). \tag{3.5}$$

By the same steps as the proof of Theorem 2.2(A_2), we obtain

$$\begin{aligned}z(x,y)&\leq G^{-1}\bigg\{G\big(a(x_0,y_0)\big)+\int_0^{\alpha(x)}\int_0^{\beta(y)}\sigma_1(s,t)\bigg[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)\\&\quad+\int_0^s\sigma_2(\tau,t)\eta\big(\psi^{-1}\big(z(\tau,t)\big)\big)\,d\tau\bigg]\,dt\,ds\bigg\}.\end{aligned}$$

We define a nonnegative and nondecreasing function $v(x,y)$ by

$$v(x,y) = G\big(a(x_0,y_0)\big) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)\bigg[\big[f(s,t)\eta\big(\psi^{-1}\big(z(s,t)\big)\big)\big] + \int_0^s \sigma_2(\tau,t)\eta\big(\psi^{-1}\big(z(\tau,t)\big)\big)\,d\tau\bigg]\,dt\,ds;$$

then $v(0,y) = v(x,0) = G(a(x_0,y_0))$,

$$z(x,y) \le G^{-1}\big[v(x,y)\big], \tag{3.6}$$

and then

$$\begin{aligned}\frac{\partial v(x,y)}{\partial x} &\le \alpha'(x)\int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\bigg[f\big(\alpha(x),t\big)\eta\big(\psi^{-1}\big(G^{-1}\big(v\big(\alpha(x),y\big)\big)\big)\big) \\ &\quad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\eta\big(\psi^{-1}\big(G^{-1}\big(v(\tau,y)\big)\big)\big)\,d\tau\bigg]\,dt \\ &\le \alpha'(x)\cdot\eta\big(\psi^{-1}\big(G^{-1}\big(v\big(\alpha(x),\beta(y)\big)\big)\big)\big)\int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\bigg[f\big(\alpha(x),t\big) \\ &\quad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\bigg]\,dt\end{aligned}$$

or

$$\begin{aligned}\frac{\frac{\partial v(x,y)}{\partial x}}{\eta(\psi^{-1}(G^{-1}(v(x,y))))} &\le \alpha'(x)\int_0^{\beta(y)} \sigma_1\big(\alpha(x),t\big)\bigg[f\big(\alpha(x),t\big) \\ &\quad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\bigg]\,dt.\end{aligned}$$

Fixing y and integrating the last inequality with respect to $s = x$ from 0 to x and using a change of variables yield the inequality

$$F\big(v(x,y)\big) \le F\big(v(0,y)\big) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t)\bigg[f(s,t) + \int_0^s \sigma_2(\tau,t)\,d\tau\bigg]\,dt\,ds$$

or

$$\begin{aligned}v(x,y) &\le F^{-1}\bigg\{F\big(G\big(a(x_0,y_0)\big)\big) + \int_0^{\alpha(x)}\int_0^{\beta(y)} \sigma_1(s,t) \\ &\quad \times \bigg[f(s,t) + \int_0^s \sigma_2(\tau,t)\,d\tau\bigg]\,dt\,ds\bigg\}.\end{aligned} \tag{3.7}$$

From (3.5)-(3.7), and since $(x_0,y_0) \in \Delta$ is chosen arbitrarily, we obtain the desired inequality (2.11). If $a(x,y) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ and subsequently let $\epsilon \to 0$. □

Proof of Corollary 2.4 (B_1) In Theorem 2.2(A_1), by letting $\psi(u) = \omega(u) = u^p$, we obtain

$$G(v) = \int_{v_0}^{v} \frac{ds}{\omega(\psi^{-1}(s))} = \int_{v_0}^{v} \frac{ds}{s} = \ln\frac{v}{v_0},$$

and hence

$$G^{-1}(v) = v_0 \exp(v), \quad v \geq v_0 > 0.$$

From equation (2.6), we obtain the inequality (2.13).

(B_2) In Theorem 2.2(A_1), by letting $\psi(u) = u^q$, $\omega(u) = u^p$ we have

$$G(v) = \int_{v_0}^{v} \frac{ds}{\omega(\psi^{-1}(s))} = \int_{v_0}^{v} \frac{ds}{s^{\frac{p}{q}}} = \frac{q}{q-p}\left(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}}\right), \quad v \geq v_0 > 0$$

and

$$G^{-1}(v) = \left\{v_0^{\frac{q-p}{q}} + \frac{q-p}{q} v\right\}^{\frac{1}{q-p}}$$

we obtain the inequality (2.15). □

Proof of Corollary 2.5 (C_1) An application of Theorem 2.2(A_2) with $\psi(u) = u^q$, $\omega(u) = u^p$, and $\eta(u) = u^r$ yields the desired inequality (2.21).

(C_2) An application of Theorem 2.2(A_3) with $\psi(u) = u^q$, $\omega(u) = u^p$, and $\eta(u) = u^r$ yields the desired inequality (2.15). □

Proof of Theorem 2.6 Suppose that $a(x,y) > 0$. Fixing an arbitrary $(x_0, y_0) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$z(x,y) = a(x_0,y_0) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\eta\bigl(u(s,t)\bigr)\left[f(s,t)\omega\bigl(u(s,t)\bigr) + \int_0^s \sigma_2(\tau,t)\,d\tau\right] dt\,ds$$

for $0 \leq x \leq x_0 \leq x_2$, $0 \leq y \leq y_0 \leq y_2$, then $z(0,y) = z(x,0) = a(x_0,y_0)$,

$$u(x,y) \leq \psi^{-1}\bigl(z(x,y)\bigr) \tag{3.8}$$

and

$$\begin{aligned}\frac{\partial z(x,y)}{\partial x} &\leq \alpha'(x)\int_0^{\beta(y)} \sigma_1\bigl(\alpha(x),t\bigr)\eta\bigl[\psi^{-1}\bigl(z(\alpha(x),t)\bigr)\bigr]\Bigl[f\bigl(\alpha(x),t\bigr)\omega\bigl(\psi^{-1}\bigl(z(\alpha(x),t)\bigr)\bigr) \\ &\quad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\Bigr] dt \\ &\leq \alpha'(x)\eta\bigl[\psi^{-1}\bigl(z(\alpha(x),\beta(y))\bigr)\bigr]\int_0^{\beta(y)} \sigma_1\bigl(\alpha(x),t\bigr)\Bigl[f\bigl(\alpha(x),t\bigr)\omega\bigl(\psi^{-1}\bigl(z(\alpha(x),t)\bigr)\bigr) \\ &\quad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\Bigr] dt,\end{aligned}$$

then

$$\begin{aligned}\frac{\frac{\partial z(x,y)}{\partial x}}{\eta[\psi^{-1}(z(x,y))]} &\leq \alpha'(x)\int_0^{\beta(y)} \sigma_1\bigl(\alpha(x),t\bigr)\Bigl[f\bigl(\alpha(x),t\bigr)\omega\bigl(\psi^{-1}\bigl(z(\alpha(x),t)\bigr)\bigr) \\ &\quad + \int_0^{\alpha(x)} \sigma_2(\tau,t)\,d\tau\Bigr] dt.\end{aligned}$$

Keeping y fixed, setting $x = s$ and integrating the last inequality with respect to s from 0 to x and making the change of variable, we obtain

$$G_1\big(z(x,y)\big) \le G_1\big(z(0,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\bigg[f(s,t)\omega\big(\psi^{-1}\big(z(s,t)\big)\big) + \int_0^s \sigma_2(\tau,t)\,d\tau\bigg]\,dt\,ds;$$

then

$$G_1\big(z(x,y)\big) \le G_1\big(a(x_0,y_0)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)\bigg[f(s,t)\omega\big(\psi^{-1}\big(z(s,t)\big)\big) + \int_0^s \sigma_2(\tau,t)\,d\tau\bigg]\,dt\,ds.$$

Since $(x_0,y_0) \in \Delta$ is chosen arbitrary, the last inequality can be restated as

$$G_1\big(z(x,y)\big) \le p_1(x,y) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\omega\big(\psi^{-1}\big(z(s,t)\big)\big)\,dt\,ds. \tag{3.9}$$

It is easy to observe that $p_1(x,y)$ is positive and nondecreasing function for all $(x,y) \in \Delta$, then an application of Lemma 2.1 to (3.9) yields the inequality

$$z(x,y) \le G_1^{-1}\left(F_1^{-1}\left[F_1\big(p_1(x,y)\big) + \int_0^{\alpha(x)} \int_0^{\beta(y)} \sigma_1(s,t)f(s,t)\,dt\,ds\right]\right). \tag{3.10}$$

From (3.10) and (3.8) we get the desired inequality (2.21).

If $a(x,y) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(x,y)$ and subsequently let $\epsilon \to 0$. □

Proof of Theorem 2.7 An application of Theorem 2.6, with $\eta(u) = u^p$ yields the desired inequality (2.26). □

Proof of Corollary 2.9 An application of Theorem 2.7 with $\psi(u(x,y)) = u^p$ to (2.28) yields the inequality (2.29); to save space we omit the details. □

4 An application

In this section, we present an application of our results to the qualitative analysis of solutions to the retarded integro differential equations. We study the boundedness of the solutions of the initial boundary value problem for partial delay integro differential equations of the form

$$D_1D_2z^q(x,y) = A\left(x,y,z\big(x-h_1(x),y-h_2(y)\big), \int_0^x B\big(s,y,z\big(s-h_1(s),y\big)\big)\,ds\right), \tag{4.1}$$

$$z(x,0) = a_1(x), \qquad z(0,y) = a_2(y), \qquad a_1(0) = a_2(0) = 0$$

for $(x,y) \in \Delta$, where $z,b \in C(\Delta,R_+)$, $A \in C(\Delta \times R^2,R)$, $B \in C(\Delta \times R,R)$ and $h_1 \in C^1(I_1,R_+)$, $h_2 \in C^1(I_2,R_+)$ are nondecreasing functions such that $h_1(x) \le x$ on I_1, $h_2(y) \le y$ on I_2, and $h_1'(x) < 1$, $h_2'(y) < 1$.

Theorem 4.1 *Assume that the functions b, A, B in* (4.1) *satisfy the conditions*

$$\bigl|a_1(x)+a_2(y)\bigr| \le a(x,y), \tag{4.2}$$

$$\bigl|A(s,t,z,u)\bigr| \le \frac{q}{q-p}\sigma_1(s,t)\bigl[f(s,t)|z|^p+|u|\bigr], \tag{4.3}$$

$$\bigl|B(\tau,t,z)\bigr| \le \sigma_2(\tau,t)|z|^p, \tag{4.4}$$

where $a(x,y)$, $\sigma_1(s,t)$, $f(s,t)$, and $\sigma_2(\tau,t)$ are as in Theorem 2.2, *$q>p>0$ are constants. If $z(x,y)$ satisfies* (4.1), *then*

$$\bigl|z(x,y)\bigr| \le \left\{p(x,y)+M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)}\overline{\sigma}_1(s,t)\overline{f}(s,t)\,dt\,ds\right\}^{\frac{1}{q-p}}, \tag{4.5}$$

where

$$\begin{aligned} p(x,y) &= \bigl(a(x,y)\bigr)^{\frac{q-p}{q}} \\ &\quad + M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)}\overline{\sigma}_1(s,t)\left(M_1\int_0^s\overline{\sigma}_2(\tau,t)\,d\tau\right)dt\,ds \end{aligned}$$

and

$$M_1=\operatorname*{Max}_{x\in I_1}\frac{1}{1-h_1'(x)},\qquad M_2=\operatorname*{Max}_{y\in I_2}\frac{1}{1-h_2'(y)}$$

and $\overline{\sigma}_1(\gamma,\xi)=\sigma_1(\gamma+h_1(s),\xi+h_2(t))$, $\overline{\sigma}_2(\mu,\xi)=\sigma_2(\mu,\xi+h_2(t))$, $\overline{f}(\gamma,\xi)=f(\gamma+h_1(s),\xi+h_2(t))$.

Proof If $z(x,y)$ is any solution of (4.1), then

$$\begin{aligned} &z^q(x,y) = a_1(x)+a_2(y), \\ &\int_0^x\int_0^y A\left(s,t,z\bigl(s-h_1(s),t-h_2(t)\bigr),\int_0^s B\bigl(\tau,t,z\bigl(\tau-h_1(\tau),t\bigr)\bigr)\,d\tau\right)dt\,ds. \end{aligned} \tag{4.6}$$

Using the conditions (4.2)-(4.4) in (4.6) we obtain

$$\begin{aligned} \bigl|z(x,y)\bigr|^q &\le a(x,y)+\frac{q-p}{q}\int_0^x\int_0^y\sigma_1(s,t)\Bigl[f(s,t)\bigl|z\bigl(s-h_1(s),t-h_2(t)\bigr)\bigr|^p \\ &\quad + \int_0^s\sigma_2(\tau,t)\bigl|z(\tau,t)\bigr|^p\,d\tau\Bigr]\,dt\,ds. \end{aligned} \tag{4.7}$$

Now making a change of variables on the right side of (4.7), $s-h_1(s)=\gamma$, $t-h_2(t)=\xi$, $x-h_1(x)=\alpha(x)$ for $x\in I_1$, $y-h_2(y)=\beta(y)$ for $y\in I_2$ we obtain the inequality

$$\begin{aligned} \bigl|z(x,y)\bigr|^q &\le a(x,y)+\frac{q-p}{q}M_1M_2\int_0^{\alpha(x)}\int_0^{\beta(y)}\overline{\sigma}_1(\gamma,\xi)\Bigl[\overline{f}(\gamma,\xi)\bigl|z(\gamma,\xi)\bigr|^p \\ &\quad + M_1\int_0^{\gamma}\overline{\sigma}_2(\mu,\xi)\bigl|z(\mu,t)\bigr|^p\,d\mu\Bigr]\,d\xi\,d\gamma. \end{aligned} \tag{4.8}$$

We can rewrite the inequality (4.8) as follows:

$$\begin{aligned}\left|z(x,y)\right|^q \le a(x,y) + \frac{q-p}{q} M_1 M_2 \int_0^{\alpha(x)} \int_0^{\beta(y)} \overline{\sigma}_1(s,t) \biggl[\overline{f}(s,t) \left|z(s,t)\right|^p \\ + M_1 \int_0^s \overline{\sigma}_2(\tau,t) \left|z(\tau,t)\right|^p \, d\tau \biggr] \, dt \, ds. \qquad (4.9)\end{aligned}$$

As an application of Corollary 2.4(B_2) to (4.9) with $u(x,y) = |z(x,y)|$ we obtain the desired inequality (4.5). □

Corollary 4.2 *If* $z(x,y)$ *satisfies the equation*

$$D_1 D_2 z^p(x,y) = A\left(x, y, z\bigl(x - h_1(x), y - h_2(y)\bigr), \int_0^x B\bigl(s, y, z\bigl(s - h_1(s), y\bigr)\bigr) \, ds\right), \qquad (4.10)$$

$$z(x,0) = a_1(x), \qquad z(0,y) = a_2(y), \qquad a_1(0) = a_2(0) = 0$$

and we suppose that the conditions (4.2)-(4.4) *are satisfied, then we have the inequality*

$$\begin{aligned}\left|z(x,y)\right|^p \le a(x,y) + M_1 M_2 \int_0^{\alpha(x)} \int_0^{\beta(y)} \overline{\sigma}_1(s,t) \biggl[\overline{f}(s,t) \left|z(s,t)\right|^p \\ + M_1 \int_0^s \overline{\sigma}_2(\tau,t) \left|z(\tau,t)\right|^p \, d\tau \biggr] \, dt \, ds, \qquad (4.11)\end{aligned}$$

then we obtain

$$\begin{aligned}\left|z(x,y)\right| \le \bigl(a(x,y)\bigr)^{\frac{1}{p}} \exp\biggl(\frac{1}{p} M_1 M_2 \int_0^{\alpha(x)} \int_0^{\beta(y)} \overline{\sigma}_1(s,t) \\ \times \biggl[\overline{f}(s,t) + M_1 \int_0^s \overline{\sigma}_2(\tau,t) \, d\tau \biggr] \, dt \, ds\biggr), \qquad (4.12)\end{aligned}$$

where $\overline{\sigma}_1, \overline{f}, \overline{\sigma}_2, M_1$, *and* M_2 *are as in Theorem* 4.1.

Proof By an application of Corollary 2.4(B_1) to (4.11) we obtain the desired inequality (4.12). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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